

AD-A055 279

ARIZONA UNIV TUCSON DEPT OF MATHEMATICS

F/6 12/1

SINGULAR PERTURBATIONS OF SEMILINEAR SECOND ORDER SYSTEMS.(U)

MAY 78 F A HOWES, R E O'MALLEY

N00014-76-C-0326

UNCLASSIFIED

TR-20

NL

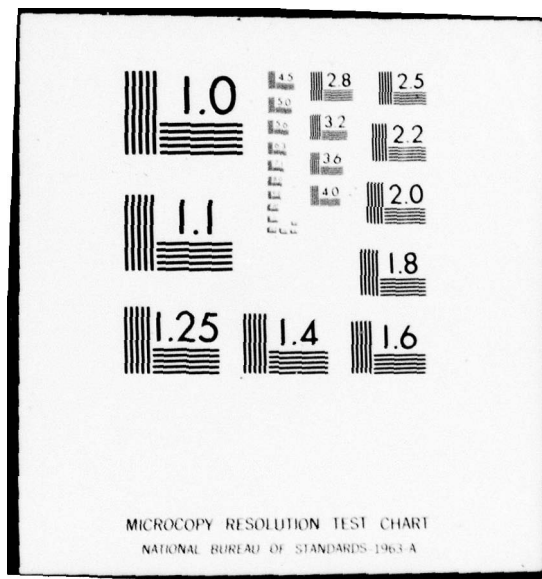
| OF |  
AD  
A055 279



END  
DATE  
FILMED

7-78

DDC



SINGULAR PERTURBATIONS OF SEMILINEAR SECOND ORDER SYSTEMS

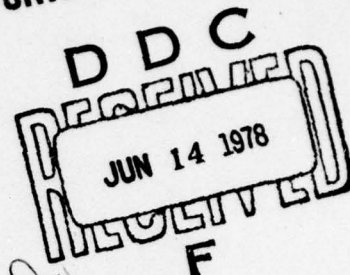
by

F. A. Howes\*  
School of Mathematics  
University of Minnesota  
Minneapolis, Minnesota 55455

and

R. E. O'Malley, Jr.\*  
Department of Mathematics and  
Program in Applied Mathematics  
University of Arizona  
Tucson, Arizona 85721

12  
FOR FURTHER TRAN



1. Problems with boundary layers at one endpoint

Many physical problems can be studied as singularly perturbed two-point vector boundary value problems of the form

$$(1) \quad \begin{cases} \epsilon y'' + f(y, t, \epsilon) y' + g(y, t, \epsilon) = 0, & 0 \leq t \leq 1 \\ y(0), y(1) \text{ prescribed} \end{cases}$$

where  $\epsilon$  is a small positive parameter (cf., e.g., Amundson (1974), Sethna and Balachandra (1976), and Cohen (1977)). Scalar problems of this form are analyzed quite thoroughly in the forthcoming memoir, Howes (1978). An enlightening case history of such analyses was given by Erdélyi (1975), and important early work includes that of Coddington and Levinson (1952) and Wasow (1956).

For simplicity, let us assume that  $f$  and  $g$  are infinitely differentiable in  $y$  and  $t$  and that they possess asymptotic power series expansions in  $\epsilon$  as  $\epsilon \rightarrow 0$ . We'll first consider the vector problem under the assumption that the reduced problem

$$(2) \quad f(u_R, t, 0) u_R' + g(u_R, t, 0) = 0, \quad u_R(1) = y(1)$$

is stable throughout  $0 \leq t \leq 1$  in the sense that  $u_R$  exists and

$$(3) \quad f(u_R(t), t, 0) > 0$$

there (i.e.,  $-f$  is a strictly stable matrix having eigenvalues with negative

Supported in part by the National Science Foundation under Grant Number MCS 76-05979 and by the Office of Naval Research under Contract Number N00014-76-C-0326.

AD A 055279

DDC FILE COPY





providing the asymptotic solution for  $t > 0$ , while the boundary layer correction  $\Pi$  has an expansion

$$(7) \quad \Pi(\tau, \epsilon) \sim \sum_{j=0}^{\infty} \Pi_j(\tau) \epsilon^j$$

whose terms all tend to zero as the stretched variable

$$(8) \quad \tau = t/\epsilon$$

tends to infinity. We would expect this solution to be unique. Under weaker smoothness assumptions on  $f$  and  $g$ , we'd have to limit the expansions to finite order approximations. For the scalar problem, Howes doesn't actually obtain higher order terms or complete boundary layer behavior, but they can easily be generated. Applying his results to the boundary value problem for the remainder terms, however, shows the asymptotic validity of the expansions so obtained.

The outer expansion (6) must provide the asymptotic solution to (1) for  $t > 0$ , since  $\Pi$  is then asymptotically negligible. Thus, the terms  $U_j$  can be successively obtained by equating coefficients in the terminal value problem

$$(9) \quad f(U(t, \epsilon), t, \epsilon) U'(t, \epsilon) + g(U(t, \epsilon), t, \epsilon) = -\epsilon U''(t, \epsilon), \quad U(1, \epsilon) = y(1).$$

Evaluating at  $\epsilon = 0$ , then, shows that  $U_0$  must satisfy the reduced problem

$$(10) \quad f(U_0(t), t, 0) U_0'(t) + g(U_0(t), t, 0) = 0, \quad U_0(1) = y(1)$$

(which has a unique solution  $u_R(t)$  under (2) and (3)). Succeeding terms  $U_j$ ,  $j > 0$ , will satisfy linear problems of the form

$$(11) \quad f(U_0(t), t, 0) U_j'(t) + f_y(U_0(t), t, 0) U_j(t) U_0'(t) + g_y(U_0(t), t, 0) U_j(t) = h_{j-1}(t), \quad U_j(1) = 0$$

where  $h_{j-1}$  is known in terms of  $t$ ,  $U_0(t)$ , ...,  $U_{j-1}(t)$ . The stability assumption (3) implies that (11) is a nonsingular initial value problem, so it also has a unique solution throughout  $0 \leq t \leq 1$ . Thus, there is no difficulty in generating the outer expansion  $U(t, \epsilon)$  with  $U(t, 0) = u_R(t)$ .

The boundary layer correction  $\Pi$  must necessarily be a decaying solution of the nonlinear initial value problem

$$(12) \quad \left\{ \frac{d^2 \Pi}{d\tau^2} + f(U(\epsilon\tau, \epsilon) + \Pi(\tau, \epsilon), \epsilon\tau, \epsilon) \frac{d\Pi}{d\tau} = -\epsilon [f(U(\epsilon\tau, \epsilon) + \Pi(\tau, \epsilon), \epsilon\tau, \epsilon) - \right.$$

$$(12) \quad \left\{ \begin{array}{l} - f(U(\epsilon\tau, \epsilon), \epsilon\tau, \epsilon) \frac{dU}{dt}(\epsilon\tau, \epsilon) + g(U(\epsilon\tau, \epsilon) + \Pi(\tau, \epsilon), \epsilon\tau, \epsilon) \\ - g(U(\epsilon\tau, \epsilon), \epsilon\tau, \epsilon) \}, \quad \tau \geq 0 \\ \Pi(0, \epsilon) = y(0) - U(0, \epsilon). \end{array} \right.$$

Thus, the leading term  $\Pi_0$  must satisfy the nonlinear problem

$$(13) \quad \frac{d^2 \Pi_0}{d\tau^2} + f(U_0(0) + \Pi_0(\tau), 0, 0) \frac{d\Pi_0}{d\tau} = 0, \quad \Pi_0(0) = y(0) - U_0(0)$$

while later terms must satisfy linear problems

$$(14) \quad \frac{d^2 \Pi_j}{d\tau^2} + f(U_0(0) + \Pi_0(\tau), 0, 0) \frac{d\Pi_j}{d\tau} + f_y(U_0(0) + \Pi_0(\tau), 0, 0) \Pi_j(\tau) \frac{d\Pi_0}{d\tau} = k_{j-1}(\tau), \quad \Pi_j(0) = -U_j(0)$$

where  $k_{j-1}$  is a linear combination of preceding terms  $\Pi_\ell$  and their derivatives  $d\Pi_\ell/d\tau$ ,  $\ell < j$ , with coefficients that are functions of  $\tau$  and  $\Pi_0(\tau)$ . The decaying solution of (13) must satisfy

$$\frac{d\Pi_0}{d\tau} + \int_0^\tau f(U_0(0) + \Pi_0, 0, 0) \frac{d\Pi_0}{d\tau} d\tau = 0$$

and, thereby, the initial value problem

$$(15) \quad \frac{d\Pi_0}{d\tau} = - \int_0^{\Pi_0(\tau)} f(U_0(0) + w, 0, 0) dw, \quad \tau \geq 0, \quad \Pi_0(0) = y(0) - u_R(0).$$

Multiplying by  $\Pi_0^T$ , the boundary layer stability condition (4) implies that

$$(16) \quad \frac{1}{2} \frac{d}{d\tau} \|\Pi_0(\tau)\|^2 = -\Pi_0^T(\tau) \int_0^{\Pi_0(\tau)} f(U_0(0) + z, 0, 0) dz < 0$$

for nonzero values of  $\Pi_0(\tau)$  satisfying  $\|\Pi_0(\tau)\| \leq \|y(0) - u_R(0)\| = \|\Pi_0(0)\|$ . Thus, our boundary layer stability implies that  $\|\Pi_0(\tau)\|$  will decrease monotonically as  $\tau$  increases until we reach the rest point  $\Pi_0(\tau) = 0$  of (15) at  $\tau = \infty$ . Ultimately,  $\Pi_0(\tau)$  will become so small that (3) (for  $t = 0$ ) implies that the eigenvalues of  $f(U_0(0) + \Pi_0(\tau), 0, 0)$  will thereafter have real parts greater than some  $\kappa > 0$  and (15) then implies that

$$(17) \quad \Pi_0(\tau) = O(e^{-\kappa\tau}),$$

i.e.,  $\Pi_0$  is exponentially decaying as  $\tau \rightarrow \infty$ . Although we can seldom explicitly integrate the nonlinear system (15), we can approximate its solution arbitrarily closely by using a successive approximations procedure on (15) (cf. Erdélyi (1964)). Knowing  $\Pi_0$ , we next integrate (14) for  $j = 1$  and then proceed term-wise. Rearranging (14) and integrating, we obtain

$$\frac{d\Pi_j}{d\tau} + f(U_0(0) + \Pi_0(\tau), 0, 0)\Pi_j + \ell_j(\tau) = 0$$

where

$$\begin{aligned} \ell_j(\tau) \equiv & \int_0^\tau \{f_y(U_0(0) + \Pi_0(r), 0, 0)[\Pi_j(r) \frac{d\Pi_0}{d\tau}(r) \\ & - \frac{d\Pi_0}{d\tau}(r)\Pi_j(r)] + k_{j-1}(r)\}dr \end{aligned}$$

is known whenever  $\Pi_j$  and  $d\Pi_0/d\tau$  commute. Thus,  $\Pi_j$  satisfies the integral equation

$$(18) \quad \Pi_j(\tau) = P(\tau)U_j(0) - \int_0^\tau P(\tau)P^{-1}(r)\ell_j(r)dr$$

where  $P(\tau)$  is the exponentially decaying fundamental matrix for the linear system

$$\frac{d\Pi}{d\tau} + f(U_0(0) + \Pi, 0, 0)\Pi = 0, \quad \tau \geq 0, \quad \Pi(0) = I.$$

In general, (18) must also be solved via successive approximations, though it directly provides the solution of (14) when the commutator  $[\Pi_j, d\Pi_0/d\tau] \equiv 0$ .

We note that the boundary layer jump  $\|\Pi_0(0)\| = \|y(0) - u_R(0)\|$  is limited by the minimum value of  $\|\tilde{\zeta}\| > 0$  such that the inner product

$$(19) \quad \tilde{\zeta}^T \int_0^{\tilde{\zeta}} f(u_R(0) + z, 0, 0)dz = 0.$$

The jump can, in practice, be quite large. It involves no restriction, for example, if  $f(z, 0, 0) > 0$  for all  $z$  since then (19) cannot ever hold for a  $\tilde{\zeta} \neq 0$ . It gives precise limits to the jumps for certain scalar problems (Howes (1977) reconsiders an example of O'Malley (1974)).

We could also consider the reduced problem

$$(20) \quad f(u_L, t, 0)u_L' + g(u_L, t, 0) = 0, \quad u_L(0) = y(0).$$

Then the stability condition (3) and the boundary layer stability condition (4) would be replaced by



$$(21) \quad f(u_L(t), t, 0) < 0$$

for  $0 \leq t \leq 1$  and the assumption that

$$(22) \quad \theta^T \int_0^1 f(u_L(1) + z, 1, 0) dz < 0$$

for all  $\theta + u_L(1)$  on paths between  $u_L(1)$  and  $y(1)$  satisfying  $0 < \|\theta\| \leq \|y(1) - u_L(1)\|$ . Nonuniform convergence of the solution to (1) would then take place near  $t = 1$ , depending on the stretched variable

$$\sigma = (1 - t)/\epsilon,$$

and the limiting solution on  $0 \leq t < 1$  would be  $u_L(t)$ . If  $f$  were nonsingular with eigenvalues having both positive and negative real parts along an appropriate solution of the reduced system, we must expect boundary layer behavior near each endpoint (cf. Harris (1973) and Ferguson (1975) for discussions of problems where  $f_y(y, t, 0) \equiv 0$ ).

## 2. Problems with boundary layers at both endpoints

Let us now consider the "twin" boundary layer problem

$$(23) \quad \begin{cases} \epsilon y'' + g(y, t, \epsilon) = 0, & 0 \leq t \leq 1 \\ y(0), y(1) \text{ prescribed} \end{cases}$$

under the assumption that  $g$  is infinitely differentiable in the region  $\tilde{D}$  of interest and that the reduced system

$$(24) \quad g(u, t, 0) = 0$$

has a smooth solution  $U_0(t)$  throughout  $0 \leq t \leq 1$  which satisfies the stability assumption

$$(25) \quad g_y(U_0(t), t, 0) < 0$$

there, i.e.,  $g_y$  is a stable matrix when evaluated along  $(U_0(t), t, 0)$ ,  $0 \leq t \leq 1$ . Motivation for this assumption is obvious if one considers the linear scalar problems with  $\epsilon y'' \pm y = 0$ , while generalized stability assumptions are sometimes appropriate and necessary (cf., e.g., Howes (1978) or consider the scalar problem with  $g = y^{2q+1}$ ). With (25), one can hope that a solution to (23) exists which converges to  $U_0(t)$  within  $(0, 1)$ . Since we won't generally have either

$U_0(0) = y(0)$  or  $U_0(1) = y(1)$ , we must expect "twin" endpoint boundary layers (i.e., regions of nonuniform convergence of thickness  $O(\sqrt{\epsilon})$  near both  $t = 0$  and  $t = 1$ ). Our previous experience (cf. Fife (1973, 1976), Yarmish (1975), O'Malley (1976), and Howes (1978)) suggests that we must add "boundary layer stability" assumptions. These generally limit the size of the boundary layer jumps  $\|y(0) - U_0(0)\|$  and  $\|y(1) - U_0(1)\|$ . They'll certainly be guaranteed if  $g_y$  remains stable throughout the boundary layer regions (cf. Kelley (1978)). Indeed, for small boundary layer jumps, the stability assumption (25) is sufficient. Under appropriate assumptions, then, we can expect to obtain an asymptotic solution to (23) in the form

$$(26) \quad y(t, \epsilon) = \tilde{U}(t, \epsilon) + \tilde{v}(\rho, \sqrt{\epsilon}) + \tilde{w}(\sigma, \sqrt{\epsilon})$$

where  $\tilde{U}$ ,  $\tilde{v}$ , and  $\tilde{w}$  all have power series expansions in their second variables and the terms of the left boundary layer correction  $\tilde{v}$  tend to zero as the stretched variable

$$(27) \quad \rho = t/\sqrt{\epsilon}$$

tends to infinity while the right boundary layer correction  $\tilde{w} \rightarrow 0$  as

$$(28) \quad \sigma = (1 - t)/\sqrt{\epsilon}$$

becomes unbounded.

The outer expansion

$$(29) \quad U(t, \epsilon) \sim \sum_{j=0}^{\infty} U_j(t) \epsilon^j$$

should therefore satisfy

$$(30) \quad \epsilon U'' + g(U, t, \epsilon) = 0, \quad 0 < t < 1$$

as a power series in  $\epsilon$  and converge to the solution  $U_0(t)$  of the reduced system (24) as  $\epsilon \rightarrow 0$ . Higher order terms in (29) must satisfy linear systems of the form

$$(31) \quad g_y(U_0, t, 0) U_j = C_{j-1}(t), \quad j \geq 1$$

where  $C_{j-1}$  is known termwise (e.g.,  $C_0 = -U_0''$ ). The stability condition (25) implies that the systems (31) are all nonsingular. Therefore successive coefficients are simply and uniquely obtained termwise. (Different roots  $U_0$  of (24)



would, of course, result in different sequences of perturbation terms  $U_j$ ,  $j > 0$ , under appropriate stability assumptions.)

According to the Borel-Ritt Theorem (cf. Wasow (1965)), there is a (non-unique) function  $\tilde{U}(t, \epsilon)$ , holomorphic in  $\epsilon$ , having the outer expansion (29). If we set

$$(32) \quad y(t, \epsilon) = \tilde{U}(t, \epsilon) + z(t, \epsilon),$$

we convert the problem (23) into the two-point problem

$$(33) \quad \epsilon z'' = h(z, t, \epsilon), \quad 0 \leq t \leq 1, \quad z(0, \epsilon) = y(0) - \tilde{U}(0, \epsilon).$$

Here

$$h(z, t, \epsilon) = -\epsilon \tilde{U}''(t, \epsilon) - g(\tilde{U}(t, \epsilon) + z, t, \epsilon)$$

satisfies

$$(34) \quad h(0, t, \epsilon) = O(\epsilon^N) \quad \text{for every integer } N \geq 0$$

since  $\epsilon \tilde{U}'' + g(\tilde{U}, t, \epsilon) = O(\epsilon^N)$ . In particular, the reduced system

$$h(z, t, 0) = 0$$

corresponding to the transformed problem (33) has the (not necessarily unique) trivial solution and the outer expansion for (33) is also trivial. Henceforth, then, we shall deal with (33) and, corresponding to (26), we shall seek an asymptotic solution of the form

$$(35) \quad z(t, \epsilon) = v(\rho, \sqrt{\epsilon}) + w(\sigma, \sqrt{\epsilon})$$

providing the needed boundary layer decay to zero within  $0 < t < 1$ . Our smoothness assumptions will be required in a domain

$$\mathcal{D}_{\epsilon_1, \delta} = \{(z, t, \epsilon): 0 \leq \|z - U_0(t)\| \leq d_\delta(t), \quad 0 \leq t \leq 1, \quad 0 \leq \epsilon \leq \epsilon_1\}$$

where  $\epsilon_1$  is a small positive number and, for any  $\delta > 0$ , we define

$$d_\delta(t) = \begin{cases} \|z(0) - U_0(0)\| + \delta, & 0 \leq t < \delta \\ \delta, & \delta \leq t \leq 1 - \delta \end{cases}$$

$$\begin{cases} \|z(1) - u_0(1)\| + \delta, & 1 - \delta < t \leq 1. \end{cases}$$

We shall determine the asymptotic behavior of  $z$  by first determining that of  $\|z\| = \sqrt{z^T z}$ . Here  $\|z\|$  satisfies the scalar problem

$$(36) \quad \epsilon \|z\|'' = [h^T(z, t, \epsilon)z + \epsilon(\|z'\|^2 - (\|z\|^2)')]/\|z\|,$$

$0 \leq t \leq 1$ , where  $\|z(0, \epsilon)\|$  and  $\|z(1, \epsilon)\|$  are prescribed.

This follows via simple calculations, namely

$$\frac{d}{dt} \|z\|^2 = 2\|z\|\|z'\| = 2(z')^T z$$

and

$$\begin{aligned} \frac{d^2}{dt^2} \|z\|^2 &= 2\|z\|\|z\|'' + 2(\|z'\|)^2 \\ &= 2(z'')^T z + 2\|z'\|^2 \end{aligned}$$

imply the differential equation for  $\|z\|$ . Further,

$$(37) \quad \|z'\|^2 \geq (\|z'\|)^2$$

since the Cauchy-Schwarz inequality  $((z')^T z)^2 \leq \|z'\|^2 \|z\|^2$  implies that  $\|z'\|^2 \geq ((z')^T z / \|z\|)^2 = (\|z'\|)^2$ . Thus, with a loss whenever  $z$  and  $z'$  are not collinear,

$$(38) \quad \epsilon \|z\|'' \geq h^T(z, t, \epsilon)z / \|z\|, \quad 0 \leq t \leq 1.$$

(Through the inequality (37), then, we eliminate the first derivative term from (38). We note that (38) is an equality for scalar problems.)

We'll now ask that for all  $(z, t, \epsilon)$  in  $\bar{D}_{\epsilon_1, \delta}$ , there exists a smooth scalar function

$$\phi(n, t, \epsilon)$$

such that

$$(39) \quad h^T(z, t, \epsilon)z \geq \phi(\|z\|, t, \epsilon)\|z\|$$

where

$$(40) \quad \phi(0, t, \epsilon) \leq 0, \quad \phi(0, t, 0) = 0, \quad \frac{\partial \phi}{\partial n}(0, t, 0) > 0$$

and

$$(41) \quad \left\{ \begin{array}{l} \int_0^\eta \phi(s, 0, 0) ds > 0 \text{ whenever } 0 < \eta \leq \|z(0, 0)\| \text{ if } z(0, 0) \neq 0 \\ \text{and} \\ \int_0^\zeta \phi(s, 1, 0) ds > 0 \text{ whenever } 0 < \zeta \leq \|z(1, 0)\| \text{ if } z(1, 0) \neq 0. \end{array} \right.$$

Existence of such a function  $\phi$  will constitute our stability hypotheses. Specifically,  $\frac{\partial \phi}{\partial n}(0, t, 0) > 0$  implies the stability of the trivial solution of the reduced system within  $(0, 1)$  while (41) implies boundary layer stability at both endpoints. Hypotheses (39)-(40) imply that

$$(42) \quad 0 \leq \|z(t, \epsilon)\| \leq m(t, \epsilon)$$

where  $m(t, \epsilon)$  satisfies the scalar two-point problem

$$(43) \quad \epsilon m'' = \phi(m, t, \epsilon), \quad 0 \leq t \leq 1, \quad m(0, \epsilon) = \|z(0, \epsilon)\|, \quad m(1, \epsilon) = \|z(1, \epsilon)\|.$$

The bounds (42) follow from the elementary theory of differential inequalities since zero is a lower solution for  $\|z\|$  and  $m$  is an upper solution (cf. Nagumo (1937), Dorr, Parter, and Shampine (1973), and Howes (1976)). Further,  $\phi(0, t, 0) = 0$  and  $\frac{\partial \phi}{\partial n}(0, t, 0) > 0$  imply that the zero solution of the reduced problem  $\phi(m, t, 0) = 0$  corresponding to (43) is stable and, according to Howes (1978), (41) is the appropriate hypothesis for the needed boundary layer stability of this solution. Indeed, the solution of (43) satisfies

$$(44) \quad m(t, \epsilon) = r_0(\rho) + s_0(\sigma) + O(\sqrt{\epsilon})$$

where  $r_0$  is the decaying solution of the boundary layer problem

$$(45) \quad \frac{d^2 r_0}{d\rho^2} = \phi(r_0, 0, 0), \quad \rho \geq 0, \quad r_0(0) = \|z(0, 0)\| = \|y(0) - u_0(0)\|$$

while  $s_0$  is the decaying solution of

$$(46) \quad \frac{d^2 s_0}{d\sigma^2} = \phi(s_0, 1, 0), \quad \sigma \geq 0, \quad s_0(0) = \|z(1, 0)\| = \|y(1) - u_0(1)\|.$$



The solutions to (45) and (46) are easily shown to exist and be unique. Multiplying (45) by  $dr_0/d\rho$ , for example, and integrating from  $\rho$  to infinity implies that

$$\frac{1}{2} \left( \frac{dr_0}{d\rho} \right)^2 = \int_0^{r_0(\rho)} \phi(s, 0, 0) ds > 0$$

(by (41)). Thus,  $r_0$  satisfies the initial value problem

$$(47) \quad \frac{dr_0}{d\rho} = -\sqrt{2 \int_0^{r_0(\rho)} \phi(s, 0, 0) ds}, \quad r_0(0) = \|y(0) - U_0(0)\|.$$

Hence,  $r_0(\rho)$  will decrease monotonically to zero as  $\rho$  increases, reaching the rest point  $r_0 = 0$  at  $\rho = \infty$ . Since  $\phi(s, 0, 0) \approx \frac{\partial \phi}{\partial n}(0, 0, 0)s$  for  $s$  small,  $\frac{\partial \phi}{\partial n}(0, 0, 0) > 0$  implies that the decay of  $r_0$  to zero is exponential as  $\rho \rightarrow \infty$ . (When  $r_0(0) = 0$ , we have  $r_0(\rho) \equiv 0$  since there is no need for a boundary layer correction.) Continuing by solving linear problems, we could obtain an asymptotic solution of (43) in the form

$$(48) \quad m(t, \epsilon) = r(\rho, \sqrt{\epsilon}) + s(\sigma, \sqrt{\epsilon}).$$

In terms of the original problem (23), our stability hypothesis (39) becomes the inequality

$$g^T(U_0(0) + z, t, \epsilon)z \leq \phi(\|z\|, t, \epsilon)\|z\|$$

where  $\phi$  satisfies (40) and (41). The expansion (44) corresponds to the expected expansion (26) for an asymptotic solution for the vector problem (23).

Now, we return to the vector boundary value problem (33) and its asymptotic solution in the form (35). Near  $t = 0$ ,  $w$  and its derivatives should be asymptotically negligible ( $\sigma$  being infinite), so (33) and (35) imply that the initial boundary layer correction  $v$  should be a decaying solution of the nonlinear initial value problem

$$(49) \quad v_{\rho\rho} = h(v, \sqrt{\epsilon}\rho, \epsilon), \quad \rho \geq 0, \quad v(0, \epsilon) = z(0, \epsilon).$$

Thus, it is natural to seek an expansion

$$(50) \quad v(\rho, \sqrt{\epsilon}) \sim \sum_{j=0}^{\infty} v_j(\rho) \epsilon^{j/2}$$

by substitution into (49). The leading term  $v_0$  must then satisfy the nonlinear

problem

$$(51) \quad \frac{d^2 v_0}{d\rho^2} = h(v_0, 0, 0), \quad \rho \geq 0, \quad v_0(0) = y(0) - U_0(0), \quad v_0 \rightarrow 0 \text{ as } \rho \rightarrow \infty.$$

Later terms  $v_j$ ,  $j \geq 1$ , must satisfy the linear problems

$$(52) \quad \begin{cases} \frac{d^2 v_j}{d\rho^2} = h_z(v_0, 0, 0)v_j + d_{j-1}(\rho), & \rho \geq 0 \\ v_j(0) = 0, & j \text{ odd}; \quad v_j(0) = -U_{j/2}(0), & j \text{ even} \\ v_j \rightarrow 0 & \text{as } \rho \rightarrow \infty \end{cases}$$

where  $d_{j-1}$  will be determined successively as an exponentially decaying vector. Since (51) and our hypothesis (39) imply that

$$\|v_0\|_{\rho\rho} \geq \phi(\|v_0\|, 0, 0), \quad \rho \geq 0, \quad \|v_0(0)\| = r_0(0),$$

we are guaranteed a decaying solution  $v_0(\rho)$  such that

$$(53) \quad 0 \leq \|v_0(\rho)\| \leq r_0(\rho), \quad \rho \geq 0$$

(and  $v_0(\rho) \equiv 0$  if  $U_0(0) = y(0)$ ). No explicit solution  $v_0$  can be provided, though an approximate solution can be obtained as usual. Introducing the matrix

$$\alpha = h_z(0, 0, 0) > 0$$

(whose eigenvalues have strictly positive real parts by our stability assumption (25)), variation of parameters can be used to express the solution of (52) in the form

$$(54) \quad v_j(\rho) = e^{-\sqrt{\alpha}\rho} \left[ v_j(0) - \int_0^\infty e^{-\sqrt{\alpha}s} F_j(s) ds \right] - \frac{1}{2} (\sqrt{\alpha})^{-1} \left[ \int_0^\rho e^{-\sqrt{\alpha}(\rho-r)} F_j(r) dr + \int_\rho^\infty e^{\sqrt{\alpha}(\rho-s)} F_j(s) ds \right]$$

where  $F_j(\rho) = [h_z(v_0(\rho), 0, 0) - \alpha]v_j(\rho) + d_{j-1}(\rho)$ . This provides the exact solution to (52) whenever  $h(v_0, 0, 0)$  is linear. Otherwise, the linear integral equation (54) must also be solved by successive approximations. In analogous fashion, we could generate the terms of the terminal boundary layer correction  $w(\sigma, \sqrt{\epsilon})$  of (35). Thus, we've formally obtained (35), which we expect is a locally unique asymptotic solution.



We note that the assumptions on  $\phi$  automatically hold if  $g_y(y, t, 0)$  or  $-h_z(z, t, 0)$  are everywhere stable. Thus, if we take

$$h_z(v, 0, 0) - \gamma I > 0$$

(i.e., positive definite) for some real  $\gamma > 0$  and all  $v$  satisfying  $0 \leq \|v\| \leq \|z(0, 0)\|$ , the mean value theorem implies that

$$h^T(z, 0, 0)z = z^T h_z(\bar{z}, 0, 0)z > \gamma \|z\|^2$$

for some "intermediate" point  $\bar{z}$ . Thus, taking  $\phi(\eta, 0, 0) = \gamma\eta$ , both  $\frac{\partial \phi}{\partial \eta}(0, 0, 0) > 0$  and  $\int_0^\eta \phi(s, 0, 0)ds > 0$  for  $0 < \eta \leq \|z(0, 0)\|$  hold.

We could also extend our discussion to systems of the form

$$\epsilon x'' = F(x, x', t, \epsilon)$$

with  $\partial F / \partial x'$  small. Thus, Kelley (1978) considered problems where  $\frac{\partial F}{\partial x} - \frac{1}{2\epsilon} \frac{\partial F}{\partial x'} \left( \frac{\partial F}{\partial x'} \right)^T > 0$ , just as Erdélyi (1968) considered scalar problems somewhat more nonlinear than semilinear.

### 3. Examples

a. A problem with an initial boundary layer

Let us consider the vector equation

$$\epsilon y'' + f(y, t, \epsilon)y' + g(y, t, \epsilon) = 0, \quad 0 \leq t \leq 1$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad f = \begin{pmatrix} y_1 & 1 \\ 1 & y_2 \end{pmatrix}, \quad \text{and} \quad g = -\begin{pmatrix} y_1 + 1 \\ y_2 + 1 \end{pmatrix}.$$

In order to have a limiting solution  $u_R$  of the two-point problem which satisfies the reduced problem

$$f(u_R, t, 0)u_R' + g(u_R, t, 0) = 0, \quad u_R(1) = y(1)$$

we must require  $u_R$  to be stable in  $0 \leq t \leq 1$ , i.e.,

$$-f(u_R(t), t, 0) < 0$$

must be a stable matrix, and we must also require boundary layer stability at

$t = 0$ , i.e., we ask that

14

$$\zeta^T \int_0^\zeta f(u_R(0) + z, 0, 0) dz > 0$$

for all  $\zeta$  such that  $0 < \|\zeta\| \leq \|y(0) - u_R(0)\|$ .

More specifically, the reduced problem has the solution

$$u_R(t) = \begin{pmatrix} t + C \\ t + D \end{pmatrix}$$

where  $C = u_{R1}(0) = -1 + y_1(1)$  and  $D = u_{R2}(0) = -1 + y_2(1)$ . Stability of  $u_R$  requires the matrix

$$\begin{pmatrix} -t - C & -1 \\ -1 & -t - D \end{pmatrix}$$

to be stable throughout  $0 \leq t \leq 1$ . This is, however, equivalent to asking that

$$C + D > 0 \quad \text{and} \quad CD > 1,$$

i.e.,

$$y_1(1)y_2(1) > y_1(1) + y_2(1) > 2.$$

Further, boundary layer stability requires that

$$\zeta^T \int_0^\zeta \begin{pmatrix} w_1 + C & 1 \\ 1 & w_2 + D \end{pmatrix} \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} > 0,$$

i.e.,

$$\zeta_1^3 + 2C\zeta_1^2 + 4\zeta_1\zeta_2 + \zeta_2^3 + 2D\zeta_2^2 > 0$$

for all  $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$  satisfying  $0 < \|\zeta\| = \|y(0) - u_R(0)\| = \sqrt{(y_1(0) - C)^2 + (y_2(0) - D)^2}$ . Our initial values  $y(0)$  are thereby restricted to a circle about  $(C, D)$  with radius less than the least norm  $\|\zeta\|$  of the nontrivial zeros of the cubic polynomial. Setting  $\zeta_2 = t\zeta_1$ , such a  $\zeta$  will satisfy

$$(1 + t^3)\zeta_1 = -2(C + 2t + Dt^2)$$

and we minimize

$$d(t) = \|\zeta\| = \sqrt{1 + t^2} |\zeta_1|.$$

(We note that the minimum for  $\zeta_1 = 0$ ,  $t = \infty$ , is 2D.) This calculus problem, then, determines an upper bound for  $\|y(0) - u_R(0)\|$ .

For  $C = D = 2$ , i.e.,  $y(1) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ , we'd obtain the minimum value 3.390 for  $d(t)$  corresponding to  $t_{\min} = -0.291$ . Thus, we're guaranteed that the limiting solution of our two-point problem is provided by  $u_R(t)$  if  $y(0)$  lies in the circle of radius 3.390 about  $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ . This is presumably a conservative estimate for the "domain of attraction" of the reduced solution  $u_R(t)$ . We expect that boundary layer stability need only hold for  $\zeta + u_R(0)$  on the actual trajectory joining  $y(0)$  and  $u_R(0)$ . Finally, we observe that this example is quite analogous to the simplest cases occurring in the analysis of solutions of the scalar problem  $\epsilon y'' + yy' - y = 0$  (cf. Cole (1968), Howes (1978), and elsewhere).

b. A problem with twin boundary layers at the endpoints

Consider the vector problem

$$\epsilon z'' = h(z, t, \epsilon), \quad 0 \leq t \leq 1$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} z_1 + z_2 - z_1^3 \\ -z_1 + z_2 - z_2^3 \end{pmatrix}.$$

Here  $U_0 = 0$  is a stable solution of the reduced problem  $h(U_0, t, 0) = 0$  since the Jacobian matrix

$$h_z(0, t, 0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has the unstable eigenvalues  $1 \pm i$ . Boundary layer stability involves the determination of a scalar function  $\phi$  such that

$$h^T(z, t, \epsilon)z \geq \phi(\|z\|, t, \epsilon)\|z\|.$$

Here

$$h^T(z, t, \epsilon)z = (z_1^2 + z_2^2) - (z_1^4 + z_2^4) \geq \|z\|^2(1 - \|z\|^2).$$

Since  $z_1^4 + z_2^4 \leq (z_1^2 + z_2^2)^2$ , so we can take

$$\phi(n, t, \epsilon) = n(1 - n^2).$$

Clearly,  $\phi(0, t, \epsilon) \leq 0$ ,  $\phi(0, t, 0) = 0$ ,  $\phi_n(0, t, 0) > 0$  and



$$\int_0^\eta \phi(s, i, 0) ds = \frac{1}{2} \eta^2 (1 - \eta^2/2) > 0 \quad \text{for } 0 < \eta < \sqrt{2}, \quad i = 0 \text{ or } 1.$$

Our preceding results, then, guarantee the existence of an asymptotic solution to the two-point problem which converges to the limiting solution  $U_0 = 0$  within  $(0, 1)$  provided the boundary values satisfy

$$\|z(0, 0)\| < \sqrt{2} \quad \text{and} \quad \|z(1, 0)\| < \sqrt{2}.$$

Indeed, we then have

$$0 \leq \|z(t, \epsilon)\| \leq m(t, \epsilon)$$

where  $m$  satisfies the scalar problem

$$\epsilon m'' = \phi(m, t, \epsilon), \quad 0 \leq t \leq 1, \quad m(i, \epsilon) = \|z(i, \epsilon)\| < \sqrt{2}, \quad i = 0 \text{ and } 1.$$

The asymptotic behavior of  $m$  follows from the scalar results of Howes (1978) and others.

### c. A problem with internal transition layers

We now consider the very special problem

$$\epsilon y'' + f(y, t, \epsilon) y' + g(y, t, \epsilon) = 0, \quad 0 \leq t \leq 1$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad f(y, t, \epsilon) = \begin{pmatrix} f_1(y_1, t, \epsilon) & f_2(y_1, y_2, t, \epsilon) \\ 0 & y_2 \end{pmatrix},$$

$$\text{and } g = \begin{pmatrix} g_1(y_1, y_2) \\ -y_2 \end{pmatrix}.$$

This system decouples into the two nonlinear scalar equations

$$\epsilon y_2'' + y_2 y_2' - y_2 = 0$$

and

$$\epsilon y_1'' + f_1(y_1, t, \epsilon) y_1' + [f_2(y_1, y_2, t, \epsilon) y_2' + g_1(y_1, y_2, t, \epsilon)] = 0.$$

If  $y_2(1) > y_2(0) + 1$  and  $-y_2(1) - 1 < y_2(0) < 1 - y_2(1)$ , it follows from Howes (1978) that the limiting solution for  $y_2$  will satisfy the reduced problem

$$u_L(u_L' - 1) = 0, \quad u_L(0) = y_2(0) \quad \text{on} \quad 0 \leq t < t^* = \frac{1}{2}(1 - y_2(1) - y_2(0))$$

and the reduced problem

$$u_R(u_R' - 1) = 0, \quad u_R(1) = y_2(1) \quad \text{on} \quad t^* < t \leq 1,$$

i.e.,

$$y_2 \rightarrow u = \begin{cases} u_L(t) = t + y_2(0), & 0 \leq t < t^* \\ u_R(t) = t + y_2(1) - 1, & t^* < t \leq 1. \end{cases}$$

Thus, the limiting solution is generally discontinuous at  $t^*$  and its derivative (which is asymptotically one elsewhere) becomes unbounded there. Indeed,  $y_2$  increases monotonically near  $t^*$  from  $u_L(t^*)$  to  $u_R(t^*) = -u_L(t^*)$ . For other relations between the boundary values  $y_2(0)$  and  $y_2(1)$ , other limiting possibilities occur (cf., e.g., Howes).

One must generally expect the transition layer at  $t^*$  in  $y_2$  to generate a corresponding discontinuity there in  $y_1$ . To simplify our discussion, however, let's assume that  $f_2(y_1, y_2, t, 0) = 0$  and attempt to apply Howes' scalar theory to the equation for  $y_1$ . Thus, consider the reduced problems

$$f_1(v_L, t, 0)v_L' + g_1(v_L, u, t, 0) = 0, \quad 0 \leq t \leq 1, \quad v_L(0) = y_1(0)$$

and

$$f_1(v_R, t, 0)v_R' + g_1(v_R, u, t, 0) = 0, \quad 0 \leq t \leq 1, \quad v_R(1) = y_1(1).$$

The limiting solution for  $y_1$  will be provided by  $v_R(t)$  if the stability condition

$$f_1(v_R(t), t, 0) > 0$$

holds throughout  $0 \leq t \leq 1$  and the boundary layer stability assumption

$$(v_R(0) - y_1(0)) \int_{\eta}^{v_R(0)} f_1(s, 0, 0) ds > 0$$

for  $\eta$  between  $v_R(0)$  and (including)  $y_1(0)$ . Similar conditions would imply that the limiting solution is  $v_L(t)$  on  $0 \leq t < 1$  with boundary layer behavior



near  $t = 1$ . If, instead, we have

$$f_1(v_R(t), t, 0) > 0 \quad \text{on} \quad t_R \leq t \leq 1$$

while

$$f_1(v_L(t), t, 0) < 0 \quad \text{on} \quad 0 \leq t \leq t_L$$

with  $t_R < t_L$ , we can expect  $y_1$  to have a limiting solution

$$y_1 \rightarrow v = \begin{cases} v_L(t), & 0 \leq t < \hat{t} \\ v_R(t), & \hat{t} < t \leq 1 \end{cases}$$

as  $\epsilon \rightarrow 0$  provided we can find a  $\hat{t}$  in  $(t_R, t_L)$  such that

$$J(\hat{t}) = 0, \quad J'(\hat{t}) \neq 0$$

for

$$J(t) = \int_{v_L(t)}^{v_R(t)} f_1(s, t, 0) ds$$

(cf. Howes (1978)). Pictorially, we will have limiting solutions  $y_2$  and  $y_1$  as shown in Figures 2 and 3.

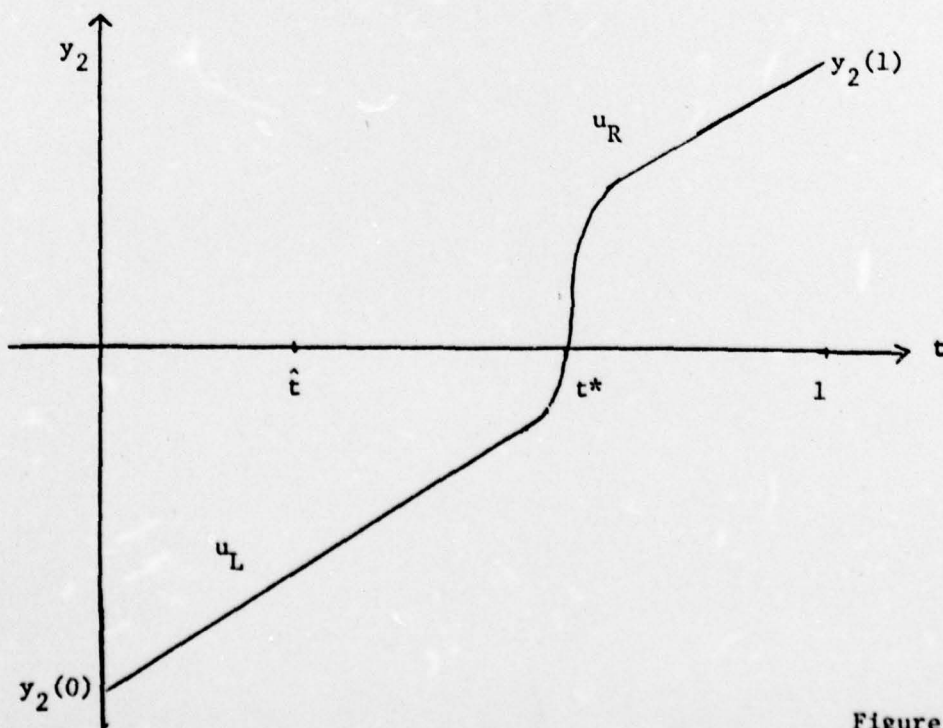


Figure 2

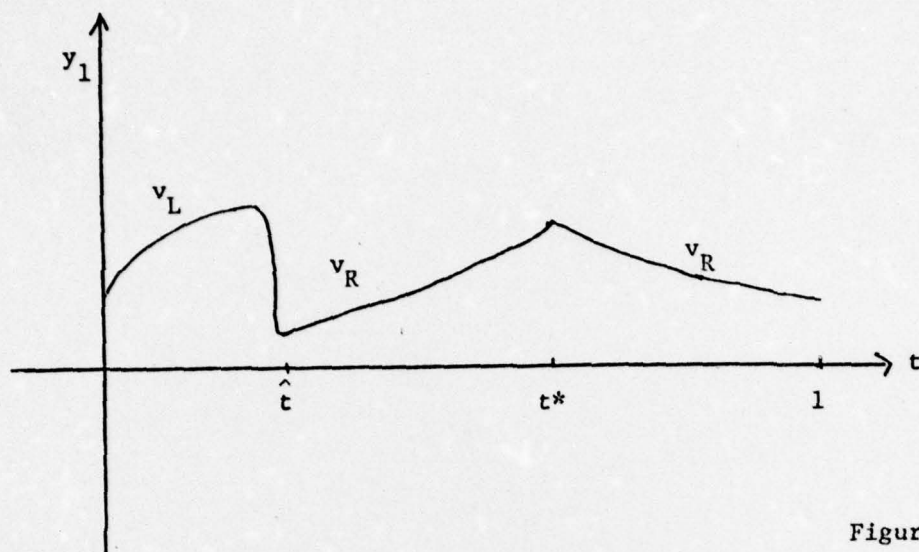


Figure 3

Note that  $y_2$  has a jump at  $\hat{t}$  and  $y_2'$  has a jump at  $t^*$ , corresponding to a Haber-Levinson crossing (cf. Howes (1978)). Much more complicated possibilities remain to be studied.

#### Acknowledgment

We wish to thank Warren Ferguson for his interest in this work and for calculating the solution to the first example.

#### References

1. N. R. Amundson, "Nonlinear problems in chemical reactor theory," SIAM-AMS Proceedings VIII (1974), 59-84.
2. E. A. Coddington and N. Levinson, "A boundary value problem for a nonlinear differential equation with a small parameter," Proc. Amer. Math. Soc. 3 (1952), 73-81.
3. D. S. Cohen, "Perturbation Theory," Lectures in Applied Mathematics 16 (1977) (American Math. Society), 61-108.
4. J. D. Cole, Perturbation Methods in Applied Mathematics, Ginn, Boston, 1968.
5. F. W. Dorr, S. V. Parter, and L. F. Shampine, "Application of the maximum principle to singular perturbation problems," SIAM Review 15 (1973), 43-88.
6. A. Erdélyi, "The integral equations of asymptotic theory," Asymptotic Solutions of Differential Equations and Their Applications (C. Wilcox, editor), Academic Press, New York, 1964, 211-229.

7. A. Erdélyi, "Approximate solutions of a nonlinear boundary value problem," Arch. Rational Mech. Anal. 29 (1968), 1-17.
8. A. Erdélyi, "A case history in singular perturbations," International Conference on Differential Equations (H. A. Antosiewicz, editor), Academic Press, New York, 1975, 266-286.
9. W. E. Ferguson, Jr., A Singularly Perturbed Linear Two-Point Boundary Value Problem, Ph.D. Dissertation, California Institute of Technology, Pasadena, 1975.
10. P. C. Fife, "Semilinear elliptic boundary value problems with small parameters," Arch. Rational Mech. Anal. 52 (1973), 205-232.
11. P. C. Fife, "Boundary and interior transition layer phenomena for pairs of second-order differential equations," J. Math. Anal. Appl. 54 (1976), 497-521.
12. W. A. Harris, Jr., "Singularly perturbed boundary value problems revisited," Lecture Notes in Math. 312 (Springer-Verlag), 1973, 54-64.
13. F. A. Howes, "Singular perturbations and differential inequalities," Memoirs Amer. Math. Soc. 168 (1976).
14. F. A. Howes, "An improved boundary layer estimate for a singularly perturbed initial value problem," unpublished manuscript, 1977.
15. F. A. Howes, "Boundary and interior layer interactions in nonlinear singular perturbation theory," Memoirs Amer. Math. Soc.
16. F. A. Howes, "Modified Haber-Levinson crossings," Trans. Amer. Math. Soc.
17. W. G. Kelley, "A nonlinear singular perturbation problem for second order systems," SIAM J. Math. Anal.
18. M. Nagumo, "Über die Differentialgleichung  $y'' = f(x, y, y')$ ," Proc. Phys. Math. Soc. Japan 19 (1937), 861-866.
19. R. E. O'Malley, Jr., Introduction to Singular Perturbations, Academic Press, New York, 1974.
20. R. E. O'Malley, Jr., "Phase-plane solutions to some singular perturbation problems," J. Math. Anal. Appl. 54 (1976), 449-466.
21. P. R. Sethna and M. B. Balachandra, "On nonlinear gyroscopic systems," Mechanics Today 3 (1976), 191-242.
22. W. Wasow, "Singular perturbation of boundary value problems for nonlinear differential equations of the second order," Comm. Pure Appl. Math. 9 (1956), 93-113.
23. W. Wasow, Asymptotic Expansions for Ordinary Differential Equations, Wiley-Interscience, New York, 1965 (Reprinted: Kreiger, Huntington, 1976).
24. J. Yarmish, "Newton's method techniques for singular perturbations," SIAM J. Math. Anal. 6 (1975), 661-680.



Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Twenty	2. GOVT ACCESSION NO. (14) TR-20	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Singular Perturbations of Semilinear Second Order Systems.		5. TYPE OF REPORT & PERIOD COVERED Technical Report
6. PERFORMING ORG. REPORT NUMBER		7. CONTRACT OR GRANT NUMBER(s)
8. AUTHOR(s) F. A./Howes and R. E./O'Malley, Jr.		9. NR0014-76-C-0326
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics University of Arizona Tucson, Arizona 85721		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR041-466
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematics Branch Office of Naval Research Arlington, Virginia		12. REPORT DATE May 1978
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES Twenty
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release. Distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Singular perturbations, asymptotic methods, boundary layers, transition layers		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Many physical problems occur as two-point boundary value problems for some systems, $\epsilon y'' + f(y, t, \epsilon)y' + g(y, t, \epsilon) = 0$ , $0 \leq t \leq 1$ , $y(0)$ , $y(1)$ pre- scribed where $\epsilon$ is a small positive parameter. Although scalar problems have been quite thoroughly analyzed in the last twenty-five years, vector problems have rarely been examined. We shall present results providing nonuniform convergence to the solution of appropriate reduced problems away from endpoint boundary layers. Possibilities involving interior transition layers will also be considered.		

DD FORM 1473

1 JAN 73

EDITION OF 1 NOV 65 IS OBSOLETE  
S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

405 313

aH